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Fundamental theorems of asset pricing for piecewise semimartingales of stochastic dimension

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Abstract This paper has two purposes. The first is to extend the notions of an n-dimensional semimartingale and its stochastic integral to a *piecewise semimartingale of stochastic dimension*. The properties of the former carry over largely intact to the latter, avoiding some of the pitfalls of infinite-dimensional stochastic integration. The second purpose is to extend two fundamental theorems of asset pricing (FTAPs): the equivalence of no free lunch with vanishing risk to the existence of an equivalent sigma-martingale measure for the price process, and the equivalence of no arbitrage of the first kind to the existence of an equivalent local martingale deflator for the set of nonnegative wealth processes.

Keywords Semimartingale · Martingale · Stochastic integration · Fundamental theorem of asset pricing · Stochastic dimension

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1 Introduction and background

1.1 Piecewise semimartingales

This paper deals with stochastic processes of finite, but not necessarily bounded, stochastic dimension. Such processes have been studied previously, for example, in the theory of branching processes and diffusions. But it does not appear that a general theory of stochastic integration has been developed for them. The setting lies between

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that of infinite-dimensional stochastic integration and the fixed-finite-dimensional case. The stronger properties of the latter carry over largely intact to the setting herein. This is one reason for our choice of extending finite-dimensional stochastic integration via localization rather than specializing infinite-dimensional stochastic integration. The other main reason for this approach is that the finite-dimensional treatment is more elementary and therefore accessible to a broader audience.

1.1.1 Related notions of stochastic integration

Stochastic integration has previously been extended to integrators taking values in infinite-dimensional spaces of varying generality [3, 4, 28, 29]. The case that is closest to that of finite-dimensional semimartingale integration is when the integrator is a sequence of semimartingales, as developed by De Donno and Pratelli [6]. Their formulation preserves many, but not all, of the nice properties of finite-dimensional stochastic integration. For example, the Ansel and Stricker theorem does not extend. A counterexample is given as Example 2 in [6] where $(H \cdot X)_t = t$, $\forall t \geq 0$, with X a local martingale.

This pathology presents a difficulty for defining admissibility of trading strategies. The notion of a limited credit line ($H \cdot X$ uniformly bounded from below) is insufficient to rule out arbitrage. Instead, more technical formulations of admissibility are necessary [5]. However, the theory of stochastic integration with respect to piecewise semimartingales, developed herein, does not have such problems. The Ansel and Sticker theorem extends as Theorem 2.15, and consequently if $H \cdot X$ is uniformly bounded from below, then H is admissible.

1.1.2 Piecewise integration

The theory of stochastic integration developed herein is a piecewise one. The integrator X takes values in $\bigcup_{n=1}^{\infty} \mathbb{R}^n$, and its integral is formed by *dissection*, that is, by localization on stochastic time intervals $]\!]\tau_{k-1}, \tau_k]\!]$ and by partitioning on the dimension of the integrator. Then stochastic integrals with respect to $X^{k,n}$, which are the $\mathbb{R}^n$-valued semimartingale "pieces" of X, may be stitched together to define $H \cdot X := H_0' X_0 + \sum_{k,n=1}^{\infty} H^{k,n} \cdot X^{k,n}$, with H' as the transpose of H.

This notion of piecewise integration provides one possible solution for how to deal with integration over dimensional changes. In $\mathbb{R}^n$-valued semimartingale stochastic integration, X is assumed to have right-continuous paths, and $\Delta(H \cdot X) = H' \Delta X$, where $\Delta X := X - X_-$, and X_- is the left-limit process of X. However, since $x - y$ is undefined when $\dim x \neq \dim y$, for $x, y \in \bigcup_{n=1}^{\infty} \mathbb{R}^n$, this approach does not immediately extend to dimensional shifts. One solution would be to adopt the convention of treating nonexistent components as if they take the value 0 (similar to the convention of $(\Delta(H \cdot X))_0 := H_0' X_0$ for stochastic integration in $\mathbb{R}^n$, as in [31]).

However, here we take a different approach, and place primary importance on preserving $H \cdot X$ as the capital gains (profits) arising from holding H shares in the assets X. This is due to the naturalness of $H \cdot X$ in this role and the centrality of capital gains to financial mathematics. For example, when a new asset enters the investable universe, its mere existence as an option for investment does not cause any portfolio values to change. So portfolio values should be *conserved* upon such an event, making the jump notion considered above incompatible with maintaining $H \cdot X$ as the capital gains process.

Instead, dimensional jumps in X are mandated to occur only as right discontinuities. This allows stochastic integration to be stopped just before each jump and resumed just afterwards. The left discontinuities, as usual, influence $H \cdot X$, while the right discontinuities serve to indicate the start of a new piece and do not affect $H \cdot X$, which remains a right-continuous process.

1.2 Fundamental theorems of asset pricing

There has been a large amount of literature on the topic of FTAPs in different settings. For a detailed history through 2006, see [9]. Here we highlight only the most relevant results pertaining to the setting of this paper.

Delbaen and Schachermayer [8] proved the equivalence of the condition "no free lunch with vanishing risk" (NFLVR) to the existence of an equivalent sigma-martingale measure (EσMM) for the price process X, when X is an $\mathbb{R}^n$-valued semimartingale.

The paper of Kabanov [17] concurrently arrived at the weaker equivalence of NFLVR with the existence of an equivalent separating measure for the set of replicable claims. However, his approach for this weaker result is more general than [8], in that the claims need not arise from stochastic integration with respect to a semimartingale. This makes Kabanov's approach well-suited for more general investigations into arbitrage, including the case herein. It is used in Sect. 3, along with Delbaen and Schachermayer's result [8] of EσMMs being dense in the space of equivalent separating measures, in order to prove Theorem 3.7, a generalization of "NFLVR $\Longleftrightarrow$ EσMM" in the piecewise setting. Specializations are proved additionally, showing that the sigma-martingale measures are local martingale measures when the price process is locally bounded, in analogy with [7].

It does not appear that there exists in the literature any sigma-martingale equivalence to a form of no approximate arbitrage in the setting of infinite-dimensional stochastic integration. A related result is proved in [2], where the setting is discrete time and the number of assets is countable, but the FTAP does not extend in its original form.

1.2.1 No arbitrage of the first kind

A different FTAP is also proved herein as Theorem 3.5, obtained by Kardaras in [22, Theorem 2.1] for the one-dimensional semimartingale case. The statement is that no arbitrage of the first kind is equivalent to the existence of an equivalent local martingale deflator (ELMD) for the set of nonnegative wealth processes. Notably, this condition does not require the closure property of passing from local martingales to martingales, so it has the virtue of being verifiable via local arguments, which is not the case for the NFLVR FTAP. The ELMD condition in $\mathbb{R}^n$-valued semimartingale markets is substantially weaker than NFLVR, allowing some arbitrages, but providing sufficient regularity for a duality-based theory of hedging and utility maximization [13, 32].

1.2.2 Large financial markets

The setting of large financial markets, introduced by Kabanov and Kramkov in [18], bears resemblance to the setting herein, but is somewhat different, since it consists of sequences of finite-dimensional market models without the dynamics of a stochastic number of assets. FTAPs relating asymptotic arbitrage to sequences of martingale measures were discovered in [19, 23–25]. Also related is the work of De Donno et al. [5], which studied super-replication and utility maximization using duality methods in a market modeled by a sequence of semimartingales, using the integration theory developed in [6].

1.3 Paper organization

This paper is organized as follows. Following this introductory section, Sect. 2 develops the notion of piecewise semimartingales of stochastic dimension. Section 2.1 introduces the notation, and Sect. 2.2 extends stochastic integration in $\mathbb{R}^n$ to these processes. The notions of martingale, local martingale and sigma-martingale are extended in Sect. 2.3. Section 3.1 provides an interpretation of a piecewise semimartingale as a price process for a market model, including providing applications for its right-discontinuities; see Remark 3.1. The "NFLVR$\Longleftrightarrow$EσMM" and "NA$_1\Longleftrightarrow$ELMD" equivalences are extended to this setting in Sect. 3.2, with corollaries given when additional regularity is present. Section 3.3 presents an explicit market based on the diverse markets of [14]. The number of assets grows without bound, NA$_1$ holds, and NFLVR fails due to existence of arbitrages, one of which is explicitly constructed.

2 Piecewise semimartingales of stochastic dimension

This section motivates and develops the notion of a piecewise semimartingale whose dimension is a finite but unbounded stochastic process, and extends stochastic integration to these processes as integrators. A natural[1] state space for such a process is $\mathbb{U} := \bigcup_{n=1}^{\infty} \mathbb{R}^n$, equipped with the topology generated by the union of the standard topologies on each $\mathbb{R}^n$. When $x, y \in \mathbb{R}^n$, then $x + y$ is defined as usual, and multiplication by a scalar is defined as usual within each $\mathbb{R}^n$. For regularity considerations, we limit our discussion to processes whose paths are composed of finitely many càdlàg pieces on all compact time intervals. Each change in dimension of the process necessitates the start of a new piece, so may only occur at a right-discontinuity.

2.1 Notation

The basic technique for manipulating $\mathbb{U}$-valued piecewise processes will be *dissection*, meaning localization on stochastic time intervals and partitioning into

[1] Another choice could be the space of sequences that have all but finitely many terms equal to 0. However, this state space lacks the dimensional information of $\mathbb{U}$. This information would need to be supplied as an auxiliary process.

$\mathbb{R}^n$-valued processes. Standard results from $\mathbb{R}^n$-valued stochastic analysis can then be applied and extended.

Indicator functions are a useful notational tool for dissecting stochastic processes, but must be reformulated to be useful in the state space $\mathbb{U}$, due to the multiplicity of zeros: $0^{(n)} \in \mathbb{R}^n$. To salvage their utility, define an additive identity element $\odot$, a topologically isolated point in $\widehat{\mathbb{U}} := \mathbb{U} \cup \{\odot\}$, distinguished from $0^{(n)} \in \mathbb{R}^n$, $n \in \mathbb{N}$. Let $\odot + x = x + \odot = x$, and $\odot x = x\odot = \odot$, for each $x \in \widehat{\mathbb{U}}$. The modified indicator is denoted by

$$\hat{\mathbf{1}}_A(t,\omega) := \begin{cases} 1 \in \mathbb{R} & \text{for } (t,\omega) \in A \subseteq [0,\infty) \times \Omega, \\ \odot & \text{otherwise.} \end{cases}$$

The usual definition of indicator is given its usual notation $\mathbf{1}_A$. To ensure that all results involving $\hat{\mathbf{1}}$ have the correct dimension for all (ω, t), even when $\hat{\mathbf{1}}$ takes the value $\odot$, it is necessary to add a zero $0^{(n)}$ of the appropriate dimension n.

All relationships among random variables hold merely almost surely (a.s.), and for stochastic processes Y and Z, $Y = Z$ means that Y and Z are indistinguishable. We use the notations $\mathbb{R}_+ := [0,\infty)$ and B' to denote the transpose of a matrix B. A process Y stopped at a random time α is denoted $Y^\alpha := (Y_{\alpha\wedge t})_{t\geq 0}$. For any process Y possessing paths with right limits at all times, Y^+ denotes the right-limit process. All $\mathbb{R}^n$-valued semimartingales are assumed to have right-continuous paths. The $\mathbb{U}$-extension of the ℓ_p-norms, referred to here as *local norms*, will also be useful. Of course, these are not norms in $\mathbb{U}$, since $\mathbb{U}$ is not even a vector space. We set

$$\begin{aligned} &|\cdot|_p : \mathbb{U} \to \mathbb{R}_+, \\ &|h|_p := \left(\sum_{i=1}^n |h_i|^p \right)^{1/p}, \quad \text{for } h \in \mathbb{R}^n,\ n \in \mathbb{N},\ p \in [1,\infty), \\ &|h|_\infty := \max_{1\leq i\leq n} |h_i|, \quad \text{for } h \in \mathbb{R}^n,\ n \in \mathbb{N}. \end{aligned}$$

2.2 Stochastic integration

Let the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, P)$ satisfy the usual conditions of $\mathcal{F}_0$ containing the P-null sets and $\mathbb{F}$ being right-continuous. Let X be a $\mathbb{U}$-valued progressive process whose paths have left and right limits at all times. Hence, $N := \dim X$ also has paths with left and right limits at all times.

Definition 2.1 A sequence of stopping times (τ_k) is called a *reset sequence* for a progressive $\mathbb{U}$-valued process X if for P-a.e. ω all of the following hold:

1. $\tau_0(\omega) = 0$, $\tau_{k-1}(\omega) \leq \tau_k(\omega)$, $\forall k \in \mathbb{N}$, and $\lim_{k\to\infty} \tau_k(\omega) = \infty$;
2. $N_t(\omega) = N^+_{\tau_{k-1}}(\omega)$ for all $t \in (\tau_{k-1}(\omega), \tau_k(\omega)]$, for each $k \in \mathbb{N}$;
3. $t \mapsto X_t(\omega)$ is right-continuous on $(\tau_{k-1}(\omega), \tau_k(\omega))$ for all $k \in \mathbb{N}$.

If X has a reset sequence, then the *minimal* one (in the sense of the fewest resets by a given time) is given by $\hat{\tau}_0 := 0$,

$$\hat{\tau}_k := \inf\{t > \hat{\tau}_{k-1} \mid X^+_t := X_{t+} \neq X_t\}, \quad k \in \mathbb{N}.$$

The existence of a reset sequence for X is a necessary regularity condition for the theory herein, whereas the choice of a reset sequence is inconsequential for most applications, a fact addressed below.

Next we extend stochastic integration to X as the integrator. When X has a discontinuity from the right, the stochastic integral will ignore it. Integration occurs from time 0 up to and including τ_1, at which point the integral is pasted together with an integral beginning just after τ_1, and so on.

For a process X as described above and a reset sequence (τ_k) for that process, *dissect* X and Ω to obtain

$$\begin{aligned} \Omega^{k,n} &:= \{\tau_{k-1} < \infty,\ N^+_{\tau_{k-1}} = n\} \subseteq \Omega, \quad \forall k, n \in \mathbb{N}, \\ X^{k,n} &:= (X^{\tau_k} - X^+_{\tau_{k-1}})\hat{\mathbf{1}}_{]\!]\tau_{k-1},\infty[\![\cap(\mathbb{R}_+\times\Omega^{k,n})} + 0^{(n)}, \quad \forall k, n \in \mathbb{N}. \end{aligned} \tag{2.1}$$

Each $X^{k,n}$ is $\mathbb{R}^n$-valued, adapted, has càdlàg paths, and is therefore optional.

Definition 2.2 A *piecewise semimartingale* X is a $\mathbb{U}$-valued progressive process having paths with left and right limits for all times and possessing a reset sequence (τ_k) such that $X^{k,n}$ is an $\mathbb{R}^n$-valued semimartingale for all $k, n \in \mathbb{N}$.

Proposition 2.5 will show that this definition and the subsequent development are not sensitive to the choice of a reset sequence. That is, if they hold for a particular reset sequence, then they hold for any. The definition allows the full generality of $\mathbb{R}^n$-valued semimartingale stochastic integration theory to be carried over to piecewise semimartingales taking values in $\mathbb{U}$.

Let X be a piecewise semimartingale and (τ_k) a reset sequence so that the $X^{k,n}$ are semimartingales, for each $k, n \in \mathbb{N}$. Let H be a $\mathbb{U}$-valued predictable process satisfying $\dim H = N = \dim X$. Dissect H to get

$$H^{k,n} := H\hat{\mathbf{1}}_{]\!]\tau_{k-1},\tau_k]\!]\cap(\mathbb{R}_+\times\Omega^{k,n})} + 0^{(n)}, \quad k, n \in \mathbb{N}. \tag{2.2}$$

Each $H^{k,n}$ is predictable, since H is predictable and $]\!]\tau_{k-1}, \tau_k]\!] \cap (\mathbb{R}_+ \times \Omega^{k,n})$ is a predictable set.

Definition 2.3 For a piecewise semimartingale X and reset sequence (τ_k), let

$$\begin{aligned} \mathcal{L}(X) &:= \{H \text{ predictable} \mid \dim H = N \text{ and } H^{k,n} \text{ is } X^{k,n}\text{-integrable},\ \forall k, n \in \mathbb{N}\}, \\ \mathcal{L}_0(X) &:= \{H \in \mathcal{L}(X) \mid H_0 = 0^{(N_0)}\}. \end{aligned}$$

For $H \in \mathcal{L}(X)$, the *stochastic integral* $H \cdot X$ is defined as

$$H \cdot X := H_0' X_0 + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(H^{k,n} \cdot X^{k,n}). \tag{2.3}$$

This is a generalization of $\mathbb{R}^n$-valued semimartingale stochastic integration, since in that case any sequence of stopping times $\tau_n \nearrow \infty$ is a reset sequence.

A $\mathbb{U}$-valued process H is called *simple predictable* for dimensional process N if it satisfies $\dim H = N$ and has the form

$$H = H_0 \hat{\mathbf{1}}_{\{0\}\times\Omega} + \sum_{i=1}^{j} H_i \hat{\mathbf{1}}_{]\!]\alpha_i,\alpha_{i+1}]\!]} + 0^{(N)},$$

where $0 = \alpha_1 \leq \cdots \leq \alpha_{j+1} < \infty$ are stopping times and H_i is $\mathcal{F}_{\alpha_i}$-measurable for $1 \leq i \leq j$. The class of such processes is denoted $\mathbb{S}(N)$, and when topologized with the topology of uniform convergence on compact time sets in probability (ucp), the resulting space is denoted $\mathbb{S}_{\text{ucp}}(N)$. Similarly, denote $\mathbb{D}_{\text{ucp}}$ as the space of adapted processes with right-continuous paths in the ucp topology.

Proposition 2.4 *If X is a piecewise semimartingale with $N = \dim X$, then*

$$X : \mathbb{S}_{\text{ucp}}(N) \to \mathbb{D}_{\text{ucp}}, \quad X(H) = H \cdot X$$

is a continuous linear operator.

Proof Let $H, H^i \in \mathbb{S}_{\text{ucp}}(N)$, $\forall i \in \mathbb{N}$, and $\lim_{i\to\infty} H^i = H$ (all limits here are ucp). Then by dissecting H^i as in (2.2) to get $H^{k,n,i}$, and interchanging stopping and ucp limits, we have $\lim_{i\to\infty} H^{k,n,i} = H^{k,n}$ for all $k, n \in \mathbb{N}$. Since the $X^{k,n}$ are semimartingales, $\lim_{i\to\infty}(H^{k,n,i} \cdot X^{k,n}) = (H^{k,n} \cdot X^{k,n})$. Thus,

$$(\forall T > 0)(\forall k_0 \in \mathbb{N})(\forall n_0 \in \mathbb{N})(\forall \varepsilon > 0)(\forall \delta > 0)(\exists i_0 \in \mathbb{N})$$

such that

$$P\Big[\sup_{0\leq t\leq T} \big|(H^{k,n,i} \cdot X^{k,n})_t - (H^{k,n} \cdot X^{k,n})_t\big| > \varepsilon\Big] < \delta, \tag{2.4}$$

whenever $k \leq k_0$, $n \leq n_0$ and $i > i_0$. For arbitrary $\varepsilon, \rho > 0$, choose k_0 sufficiently large such that $P[\tau_{k_0} \leq T] < \rho$, and n_0 large enough so that $P[\bigcup_{n>n_0,k\leq k_0} \Omega^{k,n}] < \rho$. Then pick i_0 sufficiently large such that (2.4) is satisfied for $\delta = \rho/(n_0 k_0)$. The claim is then proved by noting that for all $i \geq i_0$,

$$\begin{aligned}
&P\Big[\sup_{0\leq t\leq T} \big|(H^i \cdot X)_t - (H \cdot X)_t\big| > \varepsilon\Big] \\
&\quad\leq P\Big[\{\tau_{k_0} > T\} \cap \Big\{\sup_{0\leq t\leq T} \big|(H^i \cdot X)_t - (H \cdot X)_t\big| > \varepsilon\Big\}\Big] + P[\tau_{k_0} \leq T] \\
&\quad\leq \sum_{k=1}^{k_0}\sum_{n=1}^{n_0} P\Big[\sup_{0\leq t\leq T} \big|(H^{k,n,i} \cdot X^{k,n})_t - (H^{k,n} \cdot X^{k,n})_t\big| > \varepsilon\Big] \\
&\qquad + P\Big[\bigcup_{n>n_0,k\leq k_0} \Omega^{k,n}\Big] + P[\tau_{k_0} \leq T] \\
&\quad\leq k_0 n_0 \frac{\rho}{k_0 n_0} + \rho + \rho = 3\rho.
\end{aligned}$$

$\square$

The following proposition shows that the choice of a reset sequence (τ_k) carries no significance in the definitions of piecewise semimartingale, $\mathcal{L}(X)$ or $H \cdot X$. The proof is simple, yet tedious, and is therefore relegated to the Appendix.

Proposition 2.5 *Let X be a piecewise semimartingale and $\tilde{X}^{k,n}$ as in (2.1), but with respect to an arbitrary reset sequence $(\tilde{\tau}_k)$. Then $\tilde{X}^{k,n}$ is an $\mathbb{R}^n$-valued semimartingale for all $k, n \in \mathbb{N}$, and the class $\mathcal{L}(X)$ and the process $H \cdot X$ do not depend on the choice of reset sequence used in their definitions.*

Next we give some basic regularity properties of the stochastic integral.

Proposition 2.6 *For a piecewise semimartingale X, the following are true:*

1. *The stochastic integral $H \cdot X$ is an $\mathbb{R}$-valued semimartingale.*
2. *$\mathcal{L}(X)$ is a vector space. If $H, G \in \mathcal{L}(X)$ then $H \cdot X + G \cdot X = (H + G) \cdot X$.*
3. *If X is a piecewise semimartingale and α is a stopping time, then X^α is a piecewise semimartingale and $(X^\alpha)^{k,n} = (X^{k,n})^\alpha$, $\forall k, n \in \mathbb{N}$. Furthermore, if $H \in \mathcal{L}(X)$, then $H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N)} \in \mathcal{L}(X)$, $H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N^\alpha)} \in \mathcal{L}(X^\alpha)$, and*

$$(H \cdot X)^\alpha = (\hat{\mathbf{1}}_{[\![0,\alpha]\!]} H + 0^{(N)}) \cdot X = (\hat{\mathbf{1}}_{[\![0,\alpha]\!]} H + 0^{(N^\alpha)}) \cdot X^\alpha.$$

Proof 1. Let $\Omega^{k,0} := \{\tau_{k-1} = \infty\}$ and $\mathcal{A} := \{(k, n) \in \mathbb{N} \times \mathbb{Z}_+ \mid P[\Omega^{k,n}] > 0\}$. Define the probability measures $P^{k,n}[A] := \frac{P[A \cap \Omega^{k,n}]}{P[\Omega^{k,n}]}$, $\forall A \in \mathcal{F}$, $\forall (k, n) \in \mathcal{A}$. Then for all $(k, j) \in \mathcal{A}$, $\sum_{n=1}^\infty (H^{k,n} \cdot X^{k,n})$ is a $P^{k,j}$-semimartingale since $H^{k,n} \cdot X^{k,n} = 0$ on $(\Omega^{k,n})^c$. Then P is given by $P[A] = \sum_{n:(n,k)\in\mathcal{A}} P[\Omega^{k,n}] P^{k,n}[A]$, $\forall k \in \mathbb{N}$, $\forall A \in \mathcal{F}$, where $\sum_{n:(n,k)\in\mathcal{A}} P[\Omega^{k,n}] = 1$. As a consequence, Theorem II.3 of [31] implies that $\sum_{n=1}^\infty (H^{k,n} \cdot X^{k,n})$ is a P-semimartingale for all $k \in \mathbb{N}$. Therefore, $(H \cdot X)^{\tau_m} = \sum_{k=1}^m \sum_{n=1}^\infty (H^{k,n} \cdot X^{k,n})$ is a semimartingale for all $m \in \mathbb{N}$, since it is a finite sum of semimartingales. A process that is locally a semimartingale is a semimartingale (corollary of Theorem II.6 [31]), so we are done.

2. By (2.2), $(H + G)^{k,n} = H^{k,n} + G^{k,n}$, so the result in $\mathbb{R}^n$ proves the claim.

3. Any reset sequence (τ_k) for X is a reset sequence for X^α. Define the sets $\tilde{\Omega}^{k,n} := \{\tau_{k-1} < \infty, N^\alpha_{\tau_{k-1}+} = n\}$, and dissect X^α to get

$$\begin{aligned}
(X^\alpha)^{k,n} &= \big(X^{\tau_k \wedge \alpha} - (X^\alpha)^+_{\tau_{k-1}}\big) \hat{\mathbf{1}}_{]\!]\tau_{k-1},\infty[\![\cap (\mathbb{R}_+ \times \tilde{\Omega}^{k,n})} + 0^{(n)} \\
&= (X^{\tau_k} - X^+_{\tau_{k-1}})^\alpha \big(\hat{\mathbf{1}}_{]\!]\tau_{k-1},\infty[\![\cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(n)}\big)^\alpha \\
&= (X^{k,n})^\alpha,
\end{aligned}$$

since $\tilde{\Omega}^{k,n} \cap \{\alpha > \tau_{k-1}\} = \Omega^{k,n} \cap \{\alpha > \tau_{k-1}\}$ and $(X^\alpha)^{k,n} = (X^{k,n})^\alpha = 0^{(n)}$ on $\{\alpha \le \tau_{k-1}\}$. Thus, $(X^\alpha)^{k,n}$ is a semimartingale for all $k, n \in \mathbb{N}$, so X^α is a piecewise semimartingale.

If $H \in \mathcal{L}(X)$, then $\dim(H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N^\alpha)}) = N^\alpha$, and dissection yields

$$(H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N^\alpha)})^{k,n} = H^{k,n} \mathbf{1}_{[\![0,\alpha]\!]} \in \mathcal{L}\big((X^{k,n})^\alpha\big) = \mathcal{L}\big((X^\alpha)^{k,n}\big).$$

Therefore, $H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N^\alpha)} \in \mathcal{L}(X^\alpha)$, and so using $(X^\alpha)^{k,n} = (X^{k,n})^\alpha$ gives

$$(H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N^\alpha)}) \cdot X^\alpha = \sum_{k=1}\sum_{n=1}(H^{k,n}\mathbf{1}_{[\![0,\alpha]\!]}) \cdot (X^\alpha)^{k,n} = (H \cdot X)^\alpha.$$

The second equality is proved similarly: $(H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N^\alpha)})^{k,n} = H^{k,n}\mathbf{1}_{[\![0,\alpha]\!]}$ is in $\mathcal{L}(X^{k,n})$, and $(H\hat{\mathbf{1}}_{[\![0,\alpha]\!]} + 0^{(N)}) \cdot X = (H \cdot X)^\alpha$. □

Mémin's theorem [27, Corollary III.4] for a semimartingale Y states that the set of stochastic integrals $\{H \cdot Y \mid H \in \mathcal{L}(Y)\}$ is closed in the semimartingale topology (for details on the semimartingale topology, see [10, 27]). We extend Mémin's theorem here in preparation for the NFLVR FTAP in Theorem 3.7.

Proposition 2.7 (Mémin extension) *If X is a piecewise semimartingale, then the sets of stochastic integrals $\{H \cdot X \mid H \in \mathcal{L}(X),\ H \cdot X \geq -c\}$ are closed in the semimartingale topology for each $c > 0$, and so is $\{H \cdot X \mid H \in \mathcal{L}(X)\}$.*

Proof Denote by $\mathfrak{G}_c$ the set of stochastic integrals bounded from below by $-c$, and let Y be in the closure of $\mathfrak{G}_c$. Then there exists a sequence (H^i) such that $H^i \cdot X \in \mathfrak{G}_c$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty}(H^i \cdot X) = Y$ (all limits in this proof are assumed to be in the semimartingale topology). Y can be dissected as

$$Y = Y_0 + \sum_{k=1}^\infty\sum_{n=1}^\infty Y^{k,n}, \qquad Y^{k,n} := \mathbf{1}_{\Omega^{k,n}}(Y^{\tau_k} - Y^{\tau_{k-1}}).$$

Limits in the semimartingale topology may be interchanged with the operation of stopping a process, a fact proved at the end. For example,

$$\lim_{i\to\infty}(H^i \cdot X)^{\tau_k} = Y^{\tau_k}$$

for each $k \in \mathbb{N}$. Let $H^{k,n,i}$ be the dissection of H^i as in (2.2). Then by the definition of piecewise stochastic integration (2.3),

$$Y^{\tau_k} - Y^{\tau_{k-1}} = \lim_{i\to\infty}\big((H^i \cdot X)^{\tau_k} - (H^i \cdot X)^{\tau_{k-1}}\big) = \sum_{n=1}^\infty \mathbf{1}_{\Omega^{k,n}} \lim_{i\to\infty}(H^{k,n,i} \cdot X^{k,n}).$$

Since $\{\Omega^{k,n}\}_{n=1}^\infty \cap \{\tau_{k-1} < \infty\}$ is a partition of $\{\tau_{k-1} < \infty\}$, we may then deduce that $\lim_{i\to\infty}(H^{k,n,i} \cdot X^{k,n}) = Y^{k,n}$ for all $k, n \in \mathbb{N}$. The sets

$$\mathfrak{G}^{k,n} := \big\{H^{k,n} \cdot X^{k,n} \,\big|\, H^{k,n} \in \mathcal{L}(X^{k,n})\big\}, \quad \forall k, n \in \mathbb{N},$$

are closed in the semimartingale topology by Corollary III.4 of [27]. Therefore, there is some $\hat{H}^{k,n} \in \mathfrak{G}^{k,n}$ such that $\hat{H}^{k,n} \cdot X^{k,n} = Y^{k,n}$. Stitching the local pieces together and choosing $\hat{H}_0$ so that $\hat{H}_0' X_0 = Y_0$ provides the candidate closing integrand

$$\hat{H} := \hat{H}_0 + \sum_{k=1}^\infty\sum_{n=1}^\infty(\hat{\mathbf{1}}_{]\!]\tau_{k-1},\tau_k]\!] \cap (\mathbb{R}_+ \times \Omega^{k,n})}\hat{H}^{k,n}) + 0^{(N)}.$$

Then $\hat{H} \in \mathcal{L}(X)$ and

$$(\hat{H} \cdot X) = \hat{H}_0 X_0 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (\mathbf{1}_{\Omega^{k,n}} \hat{H}^{k,n}) \cdot X^{k,n} = Y.$$

To show that $Y \geq -c$, semimartingale convergence implies ucp convergence, which implies that $P[Y_t \geq -c,\ 0 \leq s \leq t] = 1$ for each $t \geq 0$. Therefore, $Y \in \mathfrak{G}_c$. The argument for $\{H \cdot X \mid H \in \mathcal{L}(X)\}$ is the same, but does not need this very last step.

It remains to show that stopping may be interchanged with semimartingale convergence. We use that $Y^i \to Y$ if and only if $(\xi^i \cdot Y^i)_t - (\xi^i \cdot Y)_t \to 0$ in probability for all simple, predictable, bounded sequences of processes (ξ^i), $\forall t \geq 0$ (see [10, 27]). For any stopping time α and any sequence of simple predictable bounded processes (ξ^i), $(\xi^i \mathbf{1}_{[\![0,\alpha]\!]})$ is also a sequence of simple, predictable, bounded processes. Therefore, $Y^i \to Y$ implies that $(\xi^i \cdot (Y^i)^\alpha - \xi^i \cdot Y^\alpha)_t = (\xi^i \mathbf{1}_{[\![0,\alpha]\!]} \cdot Y^i)_t - (\xi^i \mathbf{1}_{[\![0,\alpha]\!]} \cdot Y)_t \to 0$ in probability, $\forall t \geq 0$. □

2.3 Martingales

The notions of martingale and relatives may also be extended to the piecewise setting, but due to the reset feature of these processes, some care is needed.

Definition 2.8 A *piecewise martingale* is a piecewise semimartingale X such that $H \cdot X$ is a martingale whenever both $H \in \mathbb{S}(N)$ and $|H|_1$ are bounded. A *piecewise local martingale* is a process X for which there exists an increasing sequence of stopping times (ρ_i) such that $\lim_{i\to\infty} \rho_i = \infty$ and $\mathbf{1}_{\{\rho_i > 0\}} X^{\rho_i}$ is a piecewise martingale for all $i \in \mathbb{N}$. A *piecewise sigma-martingale* is a piecewise semimartingale X such that $H \cdot X$ is a sigma-martingale whenever $H \in \mathcal{L}(X)$.

It is easy to see that the definition of piecewise martingale is equivalent to the usual definition of martingale when X is an $\mathbb{R}^n$-valued semimartingale. Hence, the definition of piecewise local martingale is also consistent. The consistency of the definition of piecewise sigma-martingale follows from [16, Proposition III.6.42].

Remark 2.9 The definition of piecewise martingale could have required H to be bounded in $|\cdot|_p$ for some $p > 1$. The choice of p is somewhat arbitrary. All are equivalent for $\mathbb{R}^n$-valued semimartingales, due to the equivalence of any two norms on finite-dimensional normed spaces. But when the dimension is stochastic and unbounded, the definitions depend on the choice p. This distinction disappears under localization, as Lemma 2.10 shows. Proposition 2.13 implies that any choice of p yields the same class of piecewise local martingales. See also Corollaries 2.16 and 3.10 relating to this point.

Lemma 2.10 *Let $\mathbb{K} := \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$, where each $\mathcal{K}_n$ is a finite-dimensional normed space. Let $\mathbb{K}$ be equipped with the Borel sigma-algebra generated by the union of the norm topologies of each $\mathcal{K}_n$. Let Y be a $\mathbb{K}$-valued progressive process with associated process N^Y satisfying $N^Y = n$ whenever $Y \in \mathcal{K}_n$. Suppose that N^Y has paths that* *are left-continuous with right limits. If $\|Y\|_a$ is locally bounded for some function $\|\cdot\|_a : \mathbb{K} \to [0,\infty)$ such that the restriction of $\|\cdot\|_a$ to A is a norm whenever $A \subset \mathbb{K}$ is a vector space, then $\|Y\|_b$ is locally bounded for any $\|\cdot\|_b : \mathbb{K} \to [0,\infty)$ such that the restriction of $\|\cdot\|_b$ to A is a norm whenever $A \subset \mathbb{K}$ is a vector space.*

Proof Define the stopping times

$$\alpha_n := \inf\{t \ge 0 \mid N_t^Y > n\}, \quad n \in \mathbb{N}.$$

Since N^Y is left-continuous, $\mathbf{1}_{\{\alpha_n>0\}}(N^Y)^{\alpha_n} \le n$. For all $n \in \mathbb{N}$, the restrictions of $\|\cdot\|_a$ and $\|\cdot\|_b$ to $\mathcal{K}_n$ are equivalent. Thus for all $n \in \mathbb{N}$, there exists $c_n \in (0,\infty)$ such that

$$\left\|\mathbf{1}_{\{\alpha_n>0\}} Y^{\alpha_n}\right\|_b \le c_n \left\|\mathbf{1}_{\{\alpha_n>0\}} Y^{\alpha_n}\right\|_a.$$

Let (β_n) be a sequence of stopping times such that $\|\mathbf{1}_{\{\beta_n>0\}} Y^{\beta_n}\|_a$ is bounded, $\forall n \in \mathbb{N}$, and $\lim_{n\to\infty} \beta_n = \infty$. Then for $\rho_n := \alpha_n \wedge \beta_n$, $\|\mathbf{1}_{\{\rho_n>0\}} Y^{\rho_n}\|_b$ is bounded, $\forall n \in \mathbb{N}$. It remains to show that $\alpha_\infty := \lim_{n\to\infty} \alpha_n = \infty$.

Since $\mathbb{N}$ is discrete and N^Y has right limits, there exists an increasing sequence of random times (η_n) such that $\lim_{n\to\infty} \eta_n = \alpha_\infty$ and $N_{\eta_n}^Y = N_{\alpha_n+}^Y \ge n$ on $\{\alpha_n < \infty\}$. Thus, $\lim_{n\to\infty} N_{\eta_n}^Y = \lim_{n\to\infty} N_{\alpha_n+}^Y = \infty$ on $\{\alpha_\infty < \infty\}$. This contradicts the left-continuity of N^Y, thus $P[\alpha_\infty < \infty] = 0$. □

Lemma 2.10 invites an unambiguous extension of the notion of a *locally bounded* process taking values in a finite-dimensional normed space.

Definition 2.11 A process Y meeting the conditions of Lemma 2.10 is called a *locally bounded* process.

The following lemma will be useful in conjunction with dissection to prove some characterizations of Definition 2.8.

Lemma 2.12 *If η is a stopping time, $(C_j)_{j\in\mathbb{N}}$ is an $\mathcal{F}_\eta$-measurable partition of Ω and Y is an $\mathbb{R}^n$-valued semimartingale equal to $0^{(n)} \in \mathbb{R}^n$ on $[\![0,\eta]\!]$, then:*

1. *If $Y\mathbf{1}_{C_j}$ is a martingale for all $j \in \mathbb{N}$, and each Y_t is in L^1, then Y is a martingale.*
2. *If $Y\mathbf{1}_{C_j}$ is a local martingale for all $j \in \mathbb{N}$, then Y is a local martingale.*
3. *If $Y\mathbf{1}_{C_j}$ is a sigma-martingale for all $j \in \mathbb{N}$, then Y is a sigma-martingale.*

Proof In the first case, dominated convergence yields for $0 \le s \le t < \infty$ that

$$E[Y_t \mid \mathcal{F}_s] = \sum_{j=1}^{\infty} E[\mathbf{1}_{C_j} Y_t \mid \mathcal{F}_s] = \sum_{j=1}^{\infty} \mathbf{1}_{C_j} Y_s = Y_s.$$

For the local martingale case, for each $j \in \mathbb{N}$, let $(\rho_i^j)_{i\in\mathbb{N}}$ be a fundamental sequence for $\mathbf{1}_{C_j} Y$. Define $\rho_i := \eta \vee \sum_{j=1}^{i} \mathbf{1}_{C_j} \rho_i^j$, $\forall i \in \mathbb{N}$. Then the ρ_i are stopping times because

$$
\begin{aligned}
\{\rho_i \le t\} &= \{\rho_i \le t,\ \eta \le t\} \\
&= \bigcup_{j=1}^{\infty} (C_j \cap \{\rho_i \le t,\ \eta \le t\}) \\
&= \bigcup_{j=1}^{i} (C_j \cap \{\rho_i^j \le t,\ \eta \le t\}) \cup \bigcup_{j>i} (C_j \cap \{\eta \le t\}) \in \mathcal{F}_t,
\end{aligned}
$$

since $C_j \cap \{\eta \le t\} \in \mathcal{F}_t$ and the ρ_i^j are stopping times. Since Y is $0^{(n)}$ on $[\![0, \eta]\!]$, we have

$$
E\big[Y_t^{\rho_i} | \mathcal{F}_s\big] = \sum_{j=1}^{i} E\big[\mathbf{1}_{C_j} Y_t^{\rho_i^j} | \mathcal{F}_s\big] + 0^{(n)} = \sum_{j=1}^{i} \mathbf{1}_{C_j} Y_s^{\rho_i^j} + Y_s \sum_{j=i+1}^{\infty} \mathbf{1}_{C_j} = Y_s^{\rho_i}.
$$

If $Y\mathbf{1}_{C_j}$ is a sigma-martingale, then $Y\mathbf{1}_{C_j} = H^j \cdot M^j$ for some martingale M^j and some $H \in \mathcal{L}(M^j)$. Since Y is zero on $[\![0, \eta]\!]$, we may take M^j and H^j to be zero on this set also. By the previous property, $M := \sum_{j=1}^{\infty} \mathbf{1}_{C_j} M^j$ is a local martingale. The process $H := \sum_{j=1}^{\infty} H^j \mathbf{1}_{C_j}$ satisfies $H \in \mathcal{L}(M)$ and $H \cdot M = Y$. Therefore, Y is a sigma-martingale by [31, Theorem IV.9.88]. □

If X is a piecewise martingale, then it is an easy consequence of the definition that $X^{k,n}$ is a martingale for all $k, n \in \mathbb{N}$. However, due to the reset feature of piecewise processes, the converse is false, even if additionally $|X|_1$ is bounded.

The notions of local martingale and sigma-martingale hold globally if and only if they hold locally. This idea is made precise in the following characterizations of the piecewise notions via the properties holding on each piece.

Proposition 2.13 *A piecewise semimartingale X is a piecewise local martingale if and only if for any reset sequence, $X^{k,n}$ is a local martingale for all $k, n \in \mathbb{N}$, if and only if for some reset sequence, $X^{k,n}$ is a local martingale for all $k, n \in \mathbb{N}$.*

Proof To show that the first condition implies the second, let X be a local martingale and (ρ_i) a fundamental sequence for X. Then for $H \in \mathbb{S}(N^{\rho_i})$ and $|H|_1$ bounded, $H \cdot (\mathbf{1}_{\{\rho_i>0\}} X^{\rho_i})$ is a martingale. Let G be $\mathbb{R}^n$-valued, simple, predictable, and $|G|_1$ (or $|G|_p$, for $p \in [1, \infty]$) be bounded. Define $H := G\hat{\mathbf{1}}_{]\!]\tau_{k-1},\tau_k]\!] \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(N^{\rho_i})}$, resulting in $H \in \mathbb{S}(N^{\rho_i})$, $|H|_1$ bounded and $H \cdot (\mathbf{1}_{\{\rho_i>0\}} X^{\rho_i}) = G \cdot (\mathbf{1}_{\{\rho_i>0\}} X^{\rho_i})^{k,n}$ a martingale. Thus, $(\mathbf{1}_{\{\rho_i>0\}} X^{\rho_i})^{k,n}$ is a martingale, and since $(X^{\rho_i})^{k,n} = (X^{k,n})^{\rho_i}$ by part 3 of Proposition 2.6, then (ρ_i) is a fundamental sequence for $X^{k,n}$, which is hence a local martingale for all $k, n \in \mathbb{N}$.

The middle condition obviously implies the last condition. To show that the last implies the first, fix some reset sequence (τ_k) and suppose that $X^{k,n}$ is a local martingale for all $k, n \in \mathbb{N}$. Let $(\rho_i^{k,n})_{i\in\mathbb{N}}$ be a fundamental sequence for $X^{k,n}$, and note that since $X_0^{k,n} = 0^{(n)}$, we have $\mathbf{1}_{\{\alpha>0\}}(X^{k,n})^\alpha = (X^{k,n})^\alpha$, for any stopping time α and for all $k, n \in \mathbb{N}$. Define the stopping times $\hat{\rho}_i^k := \tau_{k-1} \vee (\sum_{n=1}^i \mathbf{1}_{\Omega^{k,n}} \rho_i^{k,n})$, $i, k \in \mathbb{N}$, and choose H to satisfy $H \in \mathbb{S}(N)$, $H_0 = 0^{(N_0)}$ and $|H|_1$ bounded. Then $(\sum_{n=1}^\infty \mathbf{1}_{\Omega^{k,n}} H^{k,n} \cdot X^{k,n})^{\hat{\rho}_i^k} = \sum_{n=1}^i \mathbf{1}_{\Omega^{k,n}} (H^{k,n} \cdot X^{k,n})^{\rho_i^{k,n}}$ is a martingale, since it is a finite sum of martingales. Then for each fixed k, $\alpha_i^k := \bigwedge_{m=1}^k \hat{\rho}_i^m$ is a fundamental sequence in $i \in \mathbb{N}$ for $(H \cdot X)^{\tau_k} = \sum_{m=1}^k \sum_{n=1}^\infty \mathbf{1}_{\Omega^{m,n}} (H^{m,n} \cdot X^{m,n})$. To get a fundamental sequence for X, for each k, let $i = i(k)$ be large enough such that $P[\alpha_{i(k)}^k < \tau_k \wedge k] < 2^{-k}$. Then $\lim_{k\to\infty} \alpha_{i(k)}^k = \infty$ a.s., and $\alpha_{i(k)}^k$ reduces $(H \cdot X)^{\tau_k}$ for each k. Therefore, each $\hat{\alpha}_p := \max(\alpha_{i(1)}^1 \wedge \tau_1, \ldots, \alpha_{i(p)}^p \wedge \tau_p)$ reduces $H \cdot X$ for all $p \in \mathbb{N}$, and $\lim_{p\to\infty} \hat{\alpha}_p = \infty$ a.s. This sequence does not depend on H, but we assumed $H_0 = 0^{(N_0)}$. To allow for H with nonzero H_0, define $\beta_p := \hat{\alpha}^p \mathbf{1}_{\{|X_0|_1 \le p\}}$. Then $\lim_{p\to\infty} \beta_p = \infty$ a.s., $H_0' X_0 \mathbf{1}_{\{\beta_p>0\}} \in L^1$, and $\mathbf{1}_{\{\beta_p>0\}}(H \cdot X)^{\beta_p}$ is a martingale. So (β_p) is fundamental for X, and therefore X is a piecewise local martingale. □

Proposition 2.14 *A piecewise semimartingale X is a piecewise sigma-martingale if and only if for all reset sequences (τ_k), $X^{k,n}$ is a sigma-martingale for all $k, n \in \mathbb{N}$, if and only if for some reset sequence (τ_k), $X^{k,n}$ is a sigma-martingale for all $k, n \in \mathbb{N}$.*

Proof Suppose that X is a piecewise sigma-martingale. Then for an arbitrary reset sequence (τ_k) and $k, n, i \in \mathbb{N}$, define the simple processes

$$G^{k,n,i} := (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i}) \hat{\mathbf{1}}_{\rrbracket \tau_{k-1}, \tau_k \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(N)} \in \mathcal{L}_0(X).$$

Then $G^{k,n,i} \cdot X = X_i^{k,n}$, which must be an $\mathbb{R}$-valued sigma-martingale by the definition of X being a piecewise sigma-martingale. Therefore, $X^{k,n} = (X_1^{k,n}, \ldots, X_n^{k,n})$ is an $\mathbb{R}^n$-valued sigma-martingale.

The middle condition obviously implies the last one. To show that the last implies the first, fix some reset sequence (τ_k) such that $X^{k,n}$ is an $\mathbb{R}^n$-valued sigma-martingale for each $k, n \in \mathbb{N}$. If $H \in \mathcal{L}(X)$, then $H^{k,n} \cdot X^{k,n}$ exists and is an $\mathbb{R}$-valued sigma-martingale, since sigma-martingales are closed under stochastic integration. Then $\sum_{n=1}^\infty H^{k,n} \cdot X^{k,n} = \sum_{n=1}^\infty \mathbf{1}_{\Omega^{k,n}} H^{k,n} \cdot X^{k,n}$ is a sigma-martingale by Lemma 2.12, and $(H \cdot X)^{\tau_k} = \sum_{j=1}^k \sum_{n=1}^\infty H^{j,n} \cdot X^{j,n}$ is a sigma-martingale, because sigma-martingales form a vector space. Finally, $H \cdot X$ is a sigma-martingale by localization [31, Theorem IV.9.88]. □

Next follows an extension of the Ansel and Stricker theorem [1] to piecewise martingales. It provides a necessary and sufficient characterization of when the local martingale property is conserved with respect to stochastic integration.

Theorem 2.15 (Ansel and Stricker extension) *Let X be a piecewise local martingale and $H \in \mathcal{L}(X)$. Then $H \cdot X$ is a local martingale if and only if there is an increasing sequence of stopping times $\alpha_j \nearrow \infty$ and a sequence (ϑ_j) of $(-\infty, 0]$-valued random variables in L^1 such that $(H' \Delta X)^{\alpha_j} \ge \vartheta_j$.*

Proof The proof of necessity relies only on the fact that $H \cdot X$ is a local martingale, so the proof of Theorem 7.3.7 in [9] suffices.

For sufficiency, suppose that there exists a sequence of stopping times $\alpha_j \nearrow \infty$ and a sequence (ϑ_j) of nonpositive random variables in L^1 such that $(H' \Delta X)^{\alpha_j} \geq \vartheta_j$. Then by dissection, $((H^{k,n})' \Delta X^{k,n})^{\alpha_j} \geq \vartheta_j$. By Proposition 2.13, $X^{k,n}$ is a local martingale, so by the usual Ansel and Stricker theorem, $H^{k,n} \cdot X^{k,n}$ is a local martingale for all $k, n \in \mathbb{N}$. Hence Lemma 2.12 implies that $\sum_{n=1}^{\infty} H^{k,n} \cdot X^{k,n}$ is a local martingale, and thus $(H \cdot X)^{\tau_m} = \sum_{k=1}^{m} \sum_{n=1}^{\infty} H^{k,n} \cdot X^{k,n}$ is a local martingale, and (τ_m) is a localizing sequence for $H \cdot X$. □

For stochastic analysis in $\mathbb{R}^n$, the set of local martingales has several other useful closure properties with respect to stochastic integration. Below are generalizations of a few of these properties to piecewise local martingales.

Corollary 2.16 *Let X be a piecewise local martingale.*

1. *If $H \in \mathcal{L}(X)$ and H is locally bounded, then $H \cdot X$ is a local martingale.*
2. *If X is continuous and $H \in \mathcal{L}(X)$, then $H \cdot X$ is a local martingale.*

Proof The processes $X^{k,n}$ are local martingales for all $k, n \in \mathbb{N}$ by Proposition 2.13. For the first case, H locally bounded implies that $H^{k,n}$ is locally bounded for all $k, n \in \mathbb{N}$. Stochastic integration in $\mathbb{R}^n$ preserves local martingality with respect to locally bounded integrands [31, Theorem IV.29]; therefore, $H^{k,n} \cdot X^{k,n}$ is a local martingale for all $k, n \in \mathbb{N}$. Using the same argument as at the end of the proof of the Ansel and Stricker extension, $H \cdot X$ is a local martingale.

In the second case, X is continuous, so $\Delta X = 0$. Hence Theorem 2.15 implies that $H \cdot X$ is a local martingale. □

3 Arbitrage in piecewise semimartingale market models

3.1 Market models with a stochastic number of assets

In this section, we specify a market model for an investable universe having a finite, but unbounded, stochastic number of assets available for investment. The process X is a piecewise semimartingale modeling the prices of the $N = \dim X$ assets. Immediately after τ_k, the market prices may reconfigure in an arbitrary way, potentially adding or removing assets.

There is a money market account B with an interest rate of zero so that $B = 1$. The process $V^{v,H}$ is the total wealth of an investor starting with initial wealth $V_0 = v$, and investing by holding $H \in \mathcal{L}_0(X)$ shares of the risky assets. All wealth processes are assumed to be *self-financing,* meaning that there exist a process $H \in \mathcal{L}_0(X)$ and an initial wealth $v \in \mathbb{R}$ such that

$$V_t^{v,H} = v + (H \cdot X)_t, \quad \forall t \geq 0.$$

Remark 3.1 It is important to be clear about what this self-financing condition implies in the model. Since $H \cdot X$ is right-continuous, it is unaffected by any discontinuities in X immediately after τ_k. Therefore, self-financing portfolios will not be affected by these jumps. This is useful for modeling certain types of events normally excluded from equity market models: the entry of new companies, the merging of several companies, and the breakup or spinoff of a company.

These events may affect portfolio values *upon their announcement*, but leave them unaffected at the point in time of their implementation. Any surprise in the announcement of such events can be manifested through a left discontinuity in X, which is passed on to V. Furthermore, there need not be any gap between announcement and implementation, since the paths of X may have both a right and left discontinuity at the same point in time. An illustrative example is when a company goes bankrupt via a jump to 0 in its stock price. This should occur via a left discontinuity, since this event should affect portfolio values through $H \cdot X$. A right discontinuity may also occur at this time, as the market transitions from n to $n-1$ assets, removing the bankrupt company as an option for investment.

We assume the standard notion of admissibility: trading strategies must have limited credit lines. That is, losses must be uniformly bounded from below.

Definition 3.2 A process H is called *admissible* for the piecewise semimartingale X if both of the following hold:

1. $H \in \mathcal{L}_0(X)$.
2. There exists a constant c such that a.s.

$$(H \cdot X)_t \geq -c, \quad \forall t \geq 0.$$

The class of *nonnegative wealth processes* is denoted by

$$\mathcal{V} := \mathcal{V}(X) := \{V^{v,H} := v + H \cdot X \mid v \in \mathbb{R}_+,\ H \in \mathcal{L}_0(X),\ V^{v,H} \geq 0\}.$$

3.2 Fundamental theorems of asset pricing

Characterizing the presence or absence of arbitrage-like notions in a market model is important for checking both the realism and viability of the model. Conversely, it is also important for discovering portfolios that may be desirable to implement in practice. In this section, we study the existence of arbitrage of the first kind and free lunch with vanishing risk, giving FTAPs for each.

The presence of arbitrage of the first kind, studied recently by Kardaras in [21, 22], may be a sufficiently strong pathology to rule out a market model for practical use. In other words, its absence NA_1 is often viewed as a minimal condition for market viability. The notion of arbitrage of the first kind has previously appeared in the literature under several different names and equivalent formulations. The name *cheap thrill* was used in [26]. The property of the set $\{V \in \mathcal{V}(X) \mid V_0 = 1\}$ being bounded in probability, previously called *BK* in [17] and *no unbounded profit with bounded risk* (*NUPBR*) in [20], was shown in [21, Proposition 1] to be equivalent to NA_1.

The condition NFLVR is stronger than NA_1. It was studied by Delbaen and Schachermayer in [7–9], and it rules out approximate arbitrage in a sense recalled below. In certain market models, such as those admitting arbitrage in stochastic portfolio theory [12, 13] and the benchmark approach [30], the flexibility of violating NFLVR while upholding NA_1 is essential.

Definition 3.3 An *arbitrage of the first kind* for X for horizon α, a finite stopping time, is an $\mathcal{F}_\alpha$-measurable random variable ψ such that $P[\psi \geq 0] = 1$, $P[\psi > 0] > 0$ and for each $v > 0$, there exists H such that $V^{v,H} := v + (H \cdot X) \in \mathcal{V}(X)$ and $V_\alpha^{v,H} \geq \psi$. If there are no arbitrages of the first kind, then NA_1 holds.

The restriction that α be a *finite* stopping time is merely due to the present setting of processes being defined on the time set $[0, \infty)$. If processes are defined on $[0, \infty]$, perhaps via limits, then this definition and subsequent results readily extend to arbitrary stopping times α. This is easy to demonstrate via the time-change $T(t) := \frac{t}{t+1}$ which maps $[0, \infty]$ to $[0, 1]$.

Definition 3.4 An *equivalent local martingale deflator* (ELMD) for $\mathcal{V}(X)$ is a strictly positive $\mathbb{R}$-valued local martingale Z such that $Z_0 = 1$ and for each $V \in \mathcal{V}(X)$, ZV is a nonnegative local martingale.

An ELMD is identical to the notion of *strict martingale density*, as in [33]. For an FTAP relating ELMDs to finitely additive, locally equivalent probability measures for $\mathbb{R}$-valued X, see [21].

When the price process is an $\mathbb{R}$-valued semimartingale, Kardaras proved in Theorem 2.1 of [22] that NA_1 is equivalent to the existence of an ELMD. The result holds for $\mathbb{R}^n$-valued semimartingales, as shown in [37], which we extend here to piecewise semimartingales of stochastic dimension. In performing the extension, it is useful to recruit the n-dimensional "market slices" running on stochastic time intervals $\rrbracket\tau_{k-1}, \tau_k\rrbracket$. These slices can be taken as markets in and of themselves, with price processes $X^{k,n}$.

Theorem 3.5 (NA_1 FTAP) *Let α be a finite stopping time.* NA_1 *holds for X for horizon α if and only if it holds for each $X^{k,n}$, $k, n \in \mathbb{N}$, for horizon α, if and only if there exists an ELMD for $\mathcal{V}(X^\alpha)$.*

Proof The strategy of the proof is to prove the implications

$$\begin{aligned}
(\mathrm{NA}_1 \text{ for } X) \quad &\Rightarrow \quad (\mathrm{NA}_1 \text{ for each } X^{k,n}) \\
&\Rightarrow \quad (\text{ELMD for } \mathcal{V}(X^\alpha)) \quad \Rightarrow \quad (\mathrm{NA}_1 \text{ for } X).
\end{aligned}$$

(NA_1 for X) $\Rightarrow$ (NA_1 for each $(X^{k,n})$): Suppose there exists an arbitrage of the first kind ψ with respect to $X^{k,n}$. Let $H^{k,n,v} \in \mathcal{L}_0(X^{k,n})$ satisfy $v + (H^{k,n,v} \cdot X^{k,n})_\alpha \geq \psi$. Define $H^v := H^{k,n,v} \hat{\mathbf{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(N)}$, which satisfies $H^v \in \mathcal{L}_0(X)$ and $H^v \cdot X = H^{k,n,v} \cdot X^{k,n}$. Therefore, for each $v > 0$, there exists $H^v \in \mathcal{L}_0(X)$ such that $H^v \cdot X \geq -v$ and $v + (H^v \cdot X)_\alpha \geq \psi$, and so ψ is an arbitrage of the first kind with respect to X.

(NA_1 for each $(X^{k,n})$) $\Rightarrow$ (ELMD for $\mathcal{V}(X^\alpha)$): By Theorem 2.1 of [22], for each $k,n \in \mathbb{N}$, there exists an ELMD $Z^{k,n}$ for $\mathcal{V}((X^{k,n})^\alpha)$. Without loss of generality (for example, substitute $Z^{k,n}$ with $\frac{(Z^{k,n})^{\tau_k}}{(Z^{k,n})^{\tau_{k-1}}}$), we may take $Z^{k,n} = 1$ on $[\![0, \tau_{k-1}]\!]$, since $X^{k,n} = 0$ here. Define $Z^k := \mathbf{1}_{\{\tau_{k-1}=\infty\}} + \sum_{n=1}^\infty \mathbf{1}_{\Omega^{k,n}} (Z^{k,n})^{\tau_k}$, $Z = \prod_{k=1}^\infty Z^k$, and for an X^α-admissible trading strategy H and $v \in \mathbb{R}$, define $Y := v + H \cdot X^\alpha$, $Y^k := Y^{\tau_k} - Y^{\tau_{k-1}}$, $Y^{k,n} := Y^k \mathbf{1}_{\Omega^{k,n}} = H^{k,n} \cdot (X^{k,n})^\alpha$. Although H being admissible for X^α does not in general imply that $H^{k,n}$ is admissible for $(X^{k,n})^\alpha$, this is fixed by further dissection of $H^{k,n}$ into pieces using the $\mathcal{F}_{\tau_{k-1}}$-measurable partition

$$\big\{\{\tau_{k-1} = \infty\},\ C_j := \{\tau_{k-1} < \infty\} \cap \{j \le (H \cdot X^\alpha)_{\tau_{k-1}} < j+1\},\ j \in \mathbb{Z}\big\}.$$

Then $\mathbf{1}_{C_j}(H^{k,n} \cdot (X^{k,n})^\alpha)$ must be uniformly bounded from below, since $H \cdot X^\alpha$ is, and $\mathbf{1}_{C_j}(H \cdot X^\alpha)_{\tau_{k-1}} < j+1$. By definition $H^{k,n} = 0^{(n)}$ on $[\![0, \tau_{k-1}]\!]$, thus $\mathbf{1}_{C_j} H^{k,n}$ is predictable, and so is $(X^{k,n})^\alpha$-admissible.

$$Z^{k,n}(\mathbf{1}_{C_j}(H^{k,n}) \cdot (X^{k,n})^\alpha) = \mathbf{1}_{C_j} Z^{k,n} Y^{k,n}$$

is therefore a local martingale for each $k,n \in \mathbb{N}$, $j \in \mathbb{Z}$. Lemma 2.12 implies that $Z^{k,n} Y^{k,n} = \sum_{j=-\infty}^\infty \mathbf{1}_{C_j} Z^{k,n} Y^{k,n}$ is a local martingale, and furthermore that $Z^k Y^k = \sum_{n=1}^\infty \mathbf{1}_{\Omega^{k,n}} Z^{k,n} Y^{k,n}$ is also a local martingale, as both processes are zero on $[\![0, \tau_{k-1}]\!]$.

We now prove by induction that $(ZY)^{\tau_k}$ is a local martingale, for each $k \in \mathbb{N}$. First, $(ZY)^{\tau_1} = Z^1(v + Y^1)$ is a local martingale by the above. Assume that $(ZY)^{\tau_{k-1}}$ and $Z^{\tau_{k-1}}$ are local martingales, and choose a fundamental sequence (ρ_j) that is a common reducing sequence for $Z^{\tau_{k-1}}$, $(ZY)^{\tau_{k-1}}$, Z^k, and $Z^k Y^k$, which can always be done by taking the minimum at each index over a reducing sequence for each. Making repeated use of $(Z^k Y^k)_t = (Z^k Y^k)_{t \vee \tau_{k-1}}$, we have for $0 \le s \le t < \infty$ that

$$\begin{aligned}
E\big[(ZY)_t^{\tau_k \wedge \rho_j} \big| \mathcal{F}_s\big] = {} & E\bigg[\bigg(\prod_{i=1}^k Z^i \sum_{m=1}^k Y^{m}{}_t^{\rho_j}\bigg)\bigg| \mathcal{F}_s\bigg] \\
&= E\Big[Z_t^{\tau_{k-1}\wedge\rho_j}\big(E\big[(Z^k Y^k)_t^{\rho_j} \big| \mathcal{F}_{s \vee \tau_{k-1}}\big] \\
&\quad + Y_t^{\tau_{k-1}\wedge\rho_j} E\big[(Z^k)_t^{\rho_j} \big| \mathcal{F}_{s\vee\tau_{k-1}}\big]\big)\Big|\mathcal{F}_s\Big] \\
&= E\Big[Z_t^{\tau_{k-1}\wedge\rho_j}\big(E\big[(Z^k Y^k)_{t\vee\tau_{k-1}}^{\rho_j} \big| \mathcal{F}_{s \vee \tau_{k-1}}\big] \\
&\quad + Y_t^{\tau_{k-1}\wedge\rho_j} E\big[(Z^k)_{t\vee\tau_{k-1}}^{\rho_j} \big| \mathcal{F}_{s\vee\tau_{k-1}}\big]\big)\Big|\mathcal{F}_s\Big] \\
&= E\big[Z_t^{\tau_{k-1}\wedge\rho_j}(Z^k)_{s\vee\tau_{k-1}}^{\rho_j}\big((Y^k)_{s\vee\tau_{k-1}}^{\rho_j} + Y_t^{\tau_{k-1}\wedge\rho_j}\big)\big|\mathcal{F}_s\big] \\
&= (Z^k)_s^{\rho_j}\big((Y^k)_s^{\rho_j} E\big[Z_t^{\tau_{k-1}\wedge\rho_j}\big|\mathcal{F}_s\big] + E\big[(ZY)_t^{\tau_{k-1}\wedge\rho_j}\big|\mathcal{F}_s\big]\big) \\
&= (Z^k)_s^{\rho_j}\big((Y^k)_s^{\rho_j} Z_s^{\tau_{k-1}\wedge\rho_j} + (ZY)_s^{\tau_{k-1}\wedge\rho_j}\big) \\
&= (ZY)_s^{\tau_k\wedge\rho_j}.
\end{aligned}$$

Thus, $(ZY)^{\tau_k}$ is a local martingale, and by choosing $Y = 1$ ($v = 1, H = 0^{(N^\alpha)}$), Z^{τ_k} can be seen to be a local martingale as well, completing the induction.

ZY and Z are local martingales by localization. Z is strictly positive, since for P-almost every ω, it is the product of finitely many strictly positive terms. Thus, Z is an ELMD for $\mathcal{V}(X^\alpha)$.

($\exists Z$ ELMD for $\mathcal{V}(X^\alpha)$) $\Rightarrow$ (NA_1 for X): The identical proof as given in Theorem 2.1 of [22] applies, since it only depends on the supermartingale properties of Z and ZV, for $V \in \mathcal{V}(X^\alpha)$. □

Theorem 3.5 can be described as holding globally if and only if it holds locally. This makes it very convenient and easy to verify in practice compared to the NFLVR notion, which can hold locally without holding globally.

Definition 3.6 For a piecewise semimartingale X and a finite stopping time α,

$$R(X) := \{(H \cdot X)_\alpha \mid H \text{ admissible}\},$$
$$C(X) := \{g \in L^\infty(\Omega, \mathcal{F}_\alpha, P) \mid g \le f \text{ for some } f \in R\}.$$

No free lunch with vanishing risk (NFLVR) with respect to X for horizon α is

$$\bar{C}(X) \cap L^\infty_+(\Omega, \mathcal{F}_\alpha, P) = \{0\},$$

where L^∞_+ denotes the a.s. bounded nonnegative random variables, and $\bar{C}(X)$ is the closure of $C(X)$ with respect to the norm topology of $L^\infty(\Omega, \mathcal{F}_\alpha, P)$.

The following FTAP characterizes NFLVR when X is a general piecewise semimartingale. When X has more regularity, we can and do say more below. The notion of an *equivalent supermartingale measure (ESMM)* for $\mathcal{V}(X)$ is used; this is a measure equivalent to P under which every $V \in \mathcal{V}(X)$ is a supermartingale.

Theorem 3.7 (NFLVR FTAP) *Let X be a piecewise semimartingale and α a finite stopping time. X satisfies NFLVR for horizon α if and only if there exists an EσMM for X^α, if and only if there exists an ESMM for $\mathcal{V}(X^\alpha)$.*

Proof We prove (NFLVR $\Longleftrightarrow$ ESMM) and then (ESMM $\Longleftrightarrow$ EσMM).

(NFLVR $\Rightarrow$ ESMM): The implication holds via the main result of Kabanov [17, Theorems 1.1 and 1.2]. To apply his result, we need that

$$\mathfrak{G}^1 := \{(H \cdot X) \mid H \text{ is predictable, } X\text{-integrable, and } H \cdot X \ge -1\}$$

is closed in the semimartingale topology, which is provided by Proposition 2.7. The other technical conditions needed are straightforward via Proposition 2.6.

(ESMM $\Rightarrow$ NFLVR): The proof given in Theorem 2.1 of [22] applies, since it only depends on the supermartingale properties of Z and ZV, for $V \in \mathcal{V}(X^\alpha)$.

(EσMM $\Rightarrow$ ESMM): If Q is an EσMM for X^α, then $H \cdot X^\alpha$ is a Q-sigma-martingale for all X^α-admissible H. It is also a Q-supermartingale since it is a sigma-martingale uniformly bounded from below [8, Theorem 5.3].

(ESMM $\Rightarrow$ EσMM): Let $\tilde{Q}$ be an ESMM for $\mathcal{V}(X^\alpha)$, equivalently for $\mathcal{V}(X)$ for horizon α, and let $\tilde{Z} := \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}P} \in \mathcal{F}_\alpha$. Define $\tilde{Z}^k := \frac{E[Z|\mathcal{F}_{\tau_k}]}{E[Z|\mathcal{F}_{\tau_{k-1}}]}$ so that $\tilde{Z} = \prod_{k=1}^\infty \tilde{Z}^k$, with convergence in L^1. If $H^{k,n}$ is $X^{k,n}$-admissible, then

$$H := H^{k,n}\hat{\mathbf{1}}_{\rrbracket \tau_{k-1},\tau_k\rrbracket \cap (\mathbb{R}_+\times\Omega^{k,n})} + 0^{(N)} \in \mathcal{L}_0(X) \quad \text{and} \quad H\cdot X = H^{k,n}\cdot X^{k,n},$$

so H is X-admissible. Hence $C(X^{k,n}) \subseteq C(X)$, and so $\tilde{Z}$ is an ESMM for $\mathcal{V}(X^{k,n})$ for horizon α. Since $H^{k,n} \in \mathcal{L}(X^{k,n})$ implies $(H^{k,n}\cdot X^{k,n})_\alpha \in \mathcal{F}_{\tau_k\wedge\alpha}$ and $(H^{k,n}\cdot X^{k,n})$ takes the value 0 on $\llbracket 0,\tau_{k-1}\rrbracket$, $\tilde{Z}^k$ is the Radon–Nikodým derivative for an ESMM for $\mathcal{V}(X^{k,n})$ for all $k,n \in \mathbb{N}$. By [8, Proposition 4.7], for all $k,n\in\mathbb{N}$ and all $\varepsilon > 0$, there exist EσMMs for $X^{k,n}$, generated by $Z_\varepsilon^{k,n}$ that satisfy $E[|Z_\varepsilon^{k,n} - \tilde{Z}^k|] < \varepsilon 2^{-n}$. The $Z_\varepsilon^{k,n}$ may be assumed to satisfy $E[Z_\varepsilon^{k,n} \mid \mathcal{F}_{\tau_{k-1}\wedge\alpha}] = 1$, since the $\tilde{Z}^k$ satisfy this. Then $Z_\varepsilon^k := \mathbf{1}_{\{\tau_{k-1}=\infty\}} + \sum_{n=1}^\infty \mathbf{1}_{\Omega^{k,n}} Z_\varepsilon^{k,n}$ generates Q^k, an EσMM for $X^{k,n}$ for all $k,n\in\mathbb{N}$, and $E[|Z_\varepsilon^k - \tilde{Z}^k|] < \varepsilon$.

The process $\hat{Z}_\varepsilon^1 := \tilde{Z}(Z_\varepsilon^1/\tilde{Z}^1) = Z_\varepsilon^1 \prod_{j=2}^\infty \tilde{Z}_j$ satisfies $\lim_{\varepsilon\to 0}\hat{Z}_\varepsilon^1 = Z$ in probability, and $E[\hat{Z}_\varepsilon^1] = 1 = E[Z]$ for all $\varepsilon > 0$. Thus, $\lim_{\varepsilon\to0}\hat{Z}_\varepsilon^1 = Z$ in L^1. If $\hat{Z}_\varepsilon^k$ is defined and $\lim_{\varepsilon\to0}\hat{Z}_\varepsilon^k = Z$ in L^1, then as above for all $\delta_k > 0$, there exists $\varepsilon_k > 0$ such that $\hat{Z}^{k+1} := \hat{Z}^k(Z_{\varepsilon_k}^{k+1}/\tilde{Z}^{k+1})$ satisfies $E[|\hat{Z}^{k+1} - \hat{Z}^k|] < \delta_k$. By induction, there exists a sequence (ε_k) such that $(\hat{Z}^k)$ is a Cauchy sequence in L^1. Hence, $Z := \tilde{Z}\prod_{k=1}^\infty (Z_{\varepsilon_k}^k/\tilde{Z}^k) = \prod_{k=1}^\infty Z_{\varepsilon_k}^k$ converges in L^1, so $E[Z] = 1$. The subscripts ε_k are now dropped. Since $(\prod_{j=1}^k Z^j, \mathcal{F}_{\tau_k\wedge\alpha})_{k\in\mathbb{N}}$ is a martingale closed by Z, the convergence is a.s. as well. For almost all ω, $\tau_k(\omega) \nearrow \infty$, so there exists k_ω with $Z^k(\omega) = 1$ for all $k > k_\omega$, and $Z(\omega) = \prod_{k=1}^{k_\omega} Z^k(\omega)$. Since $Z^k(\omega) > 0$ for all k, then $Z(\omega) > 0$. Hence, $Z = \frac{\mathrm{d}Q}{\mathrm{d}P}$ for a measure $Q \sim P$.

It remains to show that Q generated by Z is a sigma-martingale measure for X. Below, we show that $X^{k,n}$ is a Q-sigma-martingale for all $k,n\in\mathbb{N}$. Once we have this, Proposition 2.14 says that

$$H\cdot\left(\sum_{n=1}^\infty \hat{\mathbf{1}}_{\Omega^{k,n}} X^{k,n} + 0^{(N)}\right) = \sum_{n=1}^\infty H^{k,n}\cdot X^{k,n}$$

is a Q-sigma-martingale for all $H \in \mathcal{L}(X)$ and $k\in\mathbb{N}$. Then the stopped process $(H\cdot X)^{\tau_m} = H_0' X_0 + \sum_{k=1}^m \sum_{n=1}^\infty H\cdot X^{k,n}$ is a Q-sigma-martingale, so $H\cdot X$ is locally a Q-sigma-martingale, hence a Q-sigma-martingale. Thus, X is a piecewise Q-sigma-martingale by definition.

($X^{k,n}$ is a Q-sigma-martingale): Let $Z_t := E[Z \mid \mathcal{F}_t]$, $t \geq 0$, and let $M^{k,n}$ be an $\mathbb{R}^n$-valued Q^k-martingale (where $\frac{\mathrm{d}Q^k}{\mathrm{d}P} := Z^k$) such that $X^{k,n} = H^{k,n}\cdot M^{k,n}$ for some process $H^{k,n} \in \mathcal{L}(M^{k,n})$. Such processes $M^{k,n}$ and $H^{k,n}$ always exist, since $X^{k,n}$ is a Q^k-sigma-martingale. Moreover, because $X^{k,n} = 0^{(n)}$ on the complement of the set $\rrbracket\tau_{k-1},\infty\llbracket \cap (\mathbb{R}_+\times\Omega^{k,n})$, we may choose $M^{k,n} = 0^{(n)}$ there. Since $Z^{k,n}X^{k,n} = (Z^{k,n}X^{k,n})^{\tau_k}$, we may replace $M^{k,n}$ with $(M^{k,n})^{\tau_k}$. Hence, for $0 \leq s \leq t$,

$$
\begin{aligned}
E[Z_t M_t^{k,n} \mid \mathcal{F}_s] &= E\big[E[Z_t (M_t^{k,n})^{\tau_k} \mid \mathcal{F}_{s\vee\tau_k}] \,\big|\, \mathcal{F}_s\big] \\
&= E\left[Z_t^{\tau_k} M_t^{k,n} E\left[\frac{Z_t}{Z_t^{\tau_k}} \,\middle|\, \mathcal{F}_{s\vee\tau_k} \right] \middle| \mathcal{F}_s \right] \\
&= \frac{Z_s}{Z_s^{\tau_k}} E[Z_t^{\tau_k} M_t^{k,n} \mid \mathcal{F}_s] \\
&= \frac{Z_s}{Z_s^{\tau_k}} E\big[E[Z_t^{\tau_k} M_t^{k,n} \mid \mathcal{F}_{s\vee\tau_{k-1}}] \,\big|\, \mathcal{F}_s\big] \\
&= \frac{Z_s}{Z_s^{\tau_k}} E\left[\prod_{j=1}^{k-1} Z_t^j \, E\left[\sum_{m=1}^{\infty} \mathbf{1}_{\Omega^{k,m}} Z_t^{k,m} M_t^{k,n} \,\middle|\, \mathcal{F}_{s\vee\tau_{k-1}} \right] \,\middle|\, \mathcal{F}_s \right] \\
&= \frac{Z_s}{Z_s^{\tau_k}} E\left[\prod_{j=1}^{k-1} Z_t^j \, E\big[\mathbf{1}_{\Omega^{k,n}} Z_{t\vee\tau_{k-1}}^{k,n} M_{t\vee\tau_{k-1}}^{k,n} \mid \mathcal{F}_{s\vee\tau_{k-1}}\big] \,\middle|\, \mathcal{F}_s \right] \\
&= \frac{Z_s}{Z_s^{\tau_k}} Z_s^{k,n} M_s^{k,n} E\left[\prod_{j=1}^{k-1} Z_t^j \,\middle|\, \mathcal{F}_s \right] \\
&= Z_s M_s^{k,n},
\end{aligned}
$$

where we made use of $M^{k,n} = 0^{(n)}$ on the complement of $\rrbracket \tau_{k-1}, \infty\llbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})$, and of $Z_t^{k,n} M_t^{k,n} = Z_{t\vee\tau_{k-1}}^{k,n} M_{t\vee\tau_{k-1}}^{k,n}$. This establishes that $ZM^{k,n}$ is a P-martingale, so $ZX^{k,n} = Z(H^{k,n} \cdot M^{k,n})$ is a P-sigma-martingale, and thus $X^{k,n}$ is a Q-sigma-martingale. $\square$

The proof of Theorem 3.7 contains the proof of the following corollary, a generalization of [8, Proposition 4.7] and [17, Theorem 2].

Corollary 3.8 *If $\tilde{Q}$ is an ESMM for $\mathcal{V}(X)$, then for any $\varepsilon > 0$ there exists Q, an EσMM for X, such that Q and $\tilde{Q}$ are within ε of each other with respect to the total variation norm.*

When X is a bounded $\mathbb{R}^n$-valued semimartingale, any ESMM for $\mathcal{V}(X)$ is an equivalent martingale measure (EMM) for X, since $-X_i, X_i \in \mathcal{V}(X)$ for $1 \le i \le n$. However, in the piecewise setting, even if X is $\mathbb{R}^n$-valued and satisfies NFLVR, having $|X|_1$ bounded is not sufficient regularity for the existence of an EMM for X. In lieu of the $|X|_1$ boundedness assumption, we have the following sufficient condition for existence of an EMM for X.

Corollary 3.9 *If each simple, predictable H with $|H|_1$ bounded is admissible for X, then any ESMM for $\mathcal{V}(X)$ is an EMM for X. Therefore, in this case, NFLVR for X is equivalent to the existence of an EMM for X.*

Proof If $H \in \mathbb{S}(N)$ is predictable and $|H|_1$ is bounded, then H and $-H$ are both admissible. Thus, $H \cdot X$ and $-H \cdot X$ are both Q-supermartingales, hence Q-martingales. Therefore, X is a piecewise Q-martingale. □

The following corollary is the natural generalization of the FTAP for $\mathbb{R}^n$-valued locally bounded semimartingales, proved originally in [7]. Its proof is very similar to that of Corollary 3.9, with localization to obtain boundedness.

Corollary 3.10 *If X is locally bounded, then any ESMM for $\mathcal{V}(X)$ is an ELMM for X. Therefore, in this case, NFLVR for X is equivalent to the existence of an ELMM for X.*

It is straightforward that if NFLVR holds for X, then it holds for each $X^{k,n}$, but we state the result formally for completeness.

Corollary 3.11 *If X satisfies NFLVR for horizon α, a finite stopping time, then for any reset sequence (τ_k) and for any $k, n \in \mathbb{N}$, $X^{k,n}$ also does.*

Proof For $X^{k,n}$-admissible $H^{k,n}$, $H := H^{k,n}\hat{\mathbf{1}}_{]\!]\tau_{k-1},\tau_k]\!]\cap(\mathbb{R}_+\times\Omega^{k,n})} + 0^{(N)} \in \mathcal{L}(X)$ and $H \cdot X = H^{k,n} \cdot X^{k,n}$. Hence H is X-admissible and $C(X^{k,n}) \subseteq C(X)$. □

The converse of Corollary 3.11 is false in general, as the next section shows.

3.3 A piecewise semimartingale market satisfying NA_1 but not NFLVR

In this section, a market is constructed in which the number of assets grows in a stochastic, dynamic way, and is unbounded at any time $T > 0$. The market admits arbitrage relative to the market portfolio after a sufficiently long horizon $T > T^*$, and we provide an explicit strategy. This causes NFLVR to fail, but the market is easily seen to satisfy NA_1 via Theorem 3.5.

3.3.1 Premodels satisfying market diversity

The example is based on *premodels* chosen from a family of diverse equity markets originally presented in [14], and concisely summarized in [13, Sect. 9]. For our purposes here, we merely recall the relevant properties:

- The family of markets has a member $\widetilde{X}^n$ for each number of assets $n \in \{3, 4, \ldots\}$. $\widetilde{X}^n$ is an $\mathbb{R}^n$-valued Brownian diffusion with constant geometric volatility σ^n.
- $\widetilde{X}^n_i$ models the *total capitalization*, i.e., #shares×share price, of the ith company. Each behaves as a geometric Brownian motion when not the largest.
- For a $\delta \in (0, \frac{1}{2})$ of our choosing, the largest market weight satisfies via a repulsive drift-singularity the *diversity condition*

$$\widetilde{m}_t := m(\widetilde{X}_t) := \max_i \frac{\widetilde{X}^n_{i,t}}{\sum_{j=1}^n \widetilde{X}^n_{j,t}} < 1 - \delta, \quad \forall t \geq 0,\ \forall n \geq 3. \tag{3.1}$$

- By the results of [12, 13], for an arbitrary horizon, each market admits arbitrage portfolios relative to the market portfolio. The models also lack ELMMs when the filtration is the Brownian one [13, Proposition 6.2].
- The markets each satisfy a.s. square-integrability of their drift b^n, i.e.,

$$\int_0^T \left\|b^n(\widetilde{X}_t^n)\right\|^2 dt < \infty, \quad \forall T > 0.$$

So each market admits an ELMD by the usual construction, and satisfies NA_1 [13].

3.3.2 A diverse piecewise semimartingale market from the premodels

Our construction pastes the premodels together in strongly Markovian fashion by imposing a breakup of the largest company before it reaches the $1-\delta$ barrier. This breakup creates a new company, increasing the number of investable assets in the market by one. The dynamics of the market capitalization X is given recursively by the local dynamics of the nth premodel $\widetilde{X}^n$ on $\rrbracket \tau_{n-1}, \tau_n \llbracket$, where

$$\begin{aligned}
&\tau_1 := \tau_2 := 0,\\
&\tau_n := \inf\{t > \tau_{n-1} \mid m_t := m(X_t) = 1-\delta-\kappa_n\}, \quad n \geq 3,
\end{aligned}$$

and κ_n are small positive numbers, to be determined. The premodels can be chosen so that $\tau_n < \infty$ a.s. ($\widetilde{m}^n$ eventually breaches $1-\delta'$, for any $\delta' > \delta$, for any $n \in \mathbb{N}$). The initial condition is

$$X_0 = x_0 \in \{x \in (0,\infty)^3 \mid m(x) < 1-\delta-\kappa_3\}.$$

Upon reset, the starting values for $n \geq 3$ are

$$X_{i,\tau_n+} = \begin{cases} (1-\nu_n)X_{i,\tau_n} & \text{for } i = \arg\max_j X_{j,\tau_n}, \\ \nu_n(\max_j X_{j,\tau_n}) & \text{for } i = n+1, \\ X_{i,\tau_n} & \text{otherwise.} \end{cases} \tag{3.2}$$

In words, a fraction $0 < \nu_n < 1$ of the largest company is "spun off" as the new $(n+1)$st company. Total capital is conserved. For this recursive definition to define a process X on $[0,\infty)$, it is necessary that $\tau_\infty = \infty$. From the diversity property of the premodels (3.1), we have

$$\lim_{\kappa_n \to 0} P[\inf\{t > 0 \mid m(\widetilde{X}_t^n) = 1-\delta-\kappa_n\} > T] = 1, \quad \forall T > 0,\ \forall n \geq 3.$$

Thus, given any sequence (ν_n) and any $T > 0$, there exist a $\beta \in (0,1)$ and a sequence of positive reals (κ_n) such that $P[\tau_n - \tau_{n-1} > T \mid \tau_{n-1} < T] \geq \beta$ for all $n \geq 3$. This implies $\sum_{n=1}^{\infty} P[\tau_n > T \mid \tau_{n-1} < T] = \infty$, and the so-called counterpart to the Borel–Cantelli lemma gives $P[\tau_\infty \geq T] = 1$. Since $T > 0$ was arbitrary, we get $P[\tau_\infty = \infty] = 1$.

3.3.3 Arbitrage

NA_1 is satisfied for each premodel, hence for each $X^{k,n}$. Thus Theorem 3.5 implies that X satisfies NA_1. We now show that X admits portfolios that are arbitrages relative to the market portfolio by constructing one explicitly.

First, we recall the notion of functionally generated portfolios (originating in [11] with summarization in [13] and generalization in [35]) and extend it to the present setting. The *market portfolio* will be designated by

$$\mu := \frac{X}{\sum_{j=1}^N X_j}, \qquad N := \dim X.$$

Definition 3.12 A *portfolio generating function* is a function $G \in C^2(O, (0,\infty))$ such that for all $n \in \mathbb{N}$ and each $u \in O^n \subseteq \{u \in (0,1)^n \mid \sum_j u_j = 1\}$, the maps $u \mapsto u_i D_i \log G(u)$ are bounded for $1 \leq i \leq n$, where the sets O^n are open, $O := \bigcup_{n=1}^\infty O^n$ and $D_i := \frac{\partial}{\partial x_i}$. The *portfolio* π *generated by* G is

$$\pi_i = \mu_i \left(1 + D_i \log G(\mu) - \sum_{j=1}^N \mu_j D_j \log G(\mu) \right), \qquad 1 \leq i \leq N.$$

When X is an Itô process, the relative performance of a portfolio π with respect to the market portfolio μ is given by the “master formula”, Equation 11.2 of [13],

$$\log \frac{V_T^\pi}{V_T^\mu} = \log \frac{G(\mu_T)}{G(\mu_0)} + \int_0^T \mathfrak{g}_s \,\mathrm{d}s, \qquad T \geq 0,$$

where the general form of $\mathfrak{g}$ is not needed here. This is easily generalizable (for details see [34, Sect. 7.1]) to the case when X is a piecewise Itô process, by utilizing Itô's formula on $\rrbracket \tau_{k-1}, \tau_k \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})$, for $k, n \in \mathbb{N}$. Then π satisfies

$$\log \frac{V_T^\pi}{V_T^\mu} = \log \frac{G(\mu_T)}{G(\mu_0)} - \sum_{k=1}^{K_T} \big(\log G(\mu_{\tau_k +}) - \log G(\mu_{\tau_k}) \big) + \int_0^t \mathfrak{g}_s \,\mathrm{d}s. \tag{3.3}$$

We make use of the *diversity-p* family of generating functions given by

$$G_p(u) := \left(\sum_{i=1}^{\dim u} u_i^p \right)^{1/p}, \qquad u \in \mathbb{U},$$

leading to $\mathfrak{g}^{(p)} = (1-p)\gamma_\pi^*$, where the form of γ_π^* is unimportant for our purposes here. For $p \in (0,1)$, (ν_n) may be chosen (e.g., $\nu_n = n^{-2}$) such that it goes to zero sufficiently rapidly for

$$\sum_{k=1}^{\infty} \operatorname{ess\,sup}\left\{\left(\log G_p(\mu_{\tau_k+}) - \log G_p(\mu_{\tau_k})\right)\right\} < \lambda \in (0, \infty). \tag{3.4}$$

The constant volatilities σ^n are chosen to satisfy for some constants ε, $M > 0$ that

$$\varepsilon \|\xi\|^2 \le \xi \sigma^n (\sigma^n)' \xi' \le M \|\xi\|^2, \quad \forall n \ge 3. \tag{3.5}$$

This combined with the δ-diversity of the market enables Lemma 3.4 of [13] to yield $\gamma_\pi^* \ge \varepsilon(1-\delta)$. Using this along with (3.4) in (3.3) implies that

$$\log \frac{V_T^\pi}{V_T^\mu} \ge \eta - \lambda + (1-p)\varepsilon(1-\delta)T,$$

$$\text{where } \eta = \log \frac{G_p^{\min}}{G(\mu_0)}, \text{ and } G_p^{\min} = \left(\delta^p + (1-\delta)^p\right)^{1/p} \le G_p(\mu).$$

Thus, for any $T > T^* := \frac{\lambda - \eta}{(1-p)\varepsilon(1-\delta)}$, the left-hand side is strictly positive, and hence $V_T^\pi > V_T^\mu$, meaning that π is an arbitrage relative to the market.

To show nonexistence of martingale measures, suppose that there exists an EσMM Q for X. Since X is positive, Proposition 2.13 implies that X must be a Q-local martingale. Since π and μ are long-only, (3.5) yields

$$\langle \log V^\varphi \rangle_t \le tM, \quad \varphi \in \{\pi, \mu\}.$$

V^π and V^μ are then Q-martingales by Novikov's criterion. But $V_T^\pi > V_T^\mu$ implies that $V_0^\pi = E^Q[V_T^\pi] > E^Q[V_T^\mu] = V_0^\mu$, a contradiction. Hence, there does not exist an EσMM for X, and X does not satisfy NFLVR for any horizon $T > T^*$.

Remark 3.13 Concave, symmetric generating functions, like the diversity-p family, systematically underweight the largest capitalization in the market, relative to the market portfolio. This effectively bets against the future returns of the largest capitalization. Equation (3.4) results from the splitting rule (3.2), and implies that the potential logarithmic drawdown relative to the market portfolio of this bet can be bounded. If a different splitting rule were used, such as splitting capitalizations into equal halves as in [36], then there would be no such drawdown bound, and NFLVR can hold. The drawdown bound, along with the "volatility capture" guaranteed by the lower bound on γ_π^*, ensures that the bet pays off by horizon T^*. This explains why NFLVR, and indeed NA, fail.

NA_1 is a far milder condition, or put differently, A_1 is an extreme pathology. It implies an opportunity for unbounded profits with bounded risk. However, strategies like the one given entail finite intermediate-horizon drawdown, so cannot simply be leveraged up without risking negative wealth at an intermediate stage. An alternative strategy of waiting until the largest stock approaches its bound and then betting heavily against it would enable increasingly large profits, but only on sets of vanishingly small probability for a fixed horizon T.

4 Concluding remarks

The notions of $\mathbb{R}^n$-valued semimartingale, martingale and relatives are extended by localization to a piecewise semimartingale of stochastic dimension etc. The stochastic integral $H \cdot X$ is extended in kind, by stitching together pieces of stochastic integrals from $\mathbb{R}^n$-valued segments. The construction seems to preserve all of the properties of stochastic analysis in $\mathbb{R}^n$ that are local in nature. Care is needed with results relying on the boundedness of processes, as boundedness may be extended in several non-equivalent ways to $\bigcup_{n=1}^{\infty} \mathbb{R}^n$. But the notion of local boundedness extends uniquely. Some properties that are not local in nature extend as well, such as the NFLVR FTAP.

Piecewise semimartingale models open up the possibility of studying more realistic and varied market dynamics, for example, allowing companies to enter, leave, merge and split in an equity market model. The extension of fundamental theorems of asset pricing suggests that many of the results [9, 15] pertaining to superreplication and hedging that exploit $\sigma(L^\infty, L^1)$-duality should also extend to the piecewise setting. We leave investigation of these and other properties to future work.

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Appendix: Proof of Proposition 2.5

Let X be progressive and have paths with left and right limits for all times. Let (τ_k) be a reset sequence such that $X^{k,n}$ is an $\mathbb{R}^n$-valued semimartingale for each $k, n \in \mathbb{N}$. Let $\mathcal{L}(X)$ and $H \cdot X$ be defined with respect to (τ_k). Suppose that $(\tilde{\tau}_k)$ is an arbitrary reset sequence for X, with corresponding $\tilde{X}^{k,n}$, $\tilde{H}^{k,n}$, $\tilde{\Omega}^{k,n}$, $\tilde{\mathcal{L}}(X)$. For $H \in \mathcal{L}_0(X)$, we have

$$
\begin{aligned}
H \cdot X &= \sum_{j=1}^{\infty} \big((H \cdot X)^{\tilde{\tau}_j} - (H \cdot X)^{\tilde{\tau}_{j-1}}\big) \\
&= \sum_{j,k,n=1}^{\infty} \big((H^{k,n} \cdot X^{k,n})^{\tilde{\tau}_j} - (H^{k,n} \cdot X^{k,n})^{\tilde{\tau}_{j-1}}\big) \\
&= \sum_{j,k,n=1}^{\infty} (H^{k,n} \mathbf{1}_{\rrbracket \tilde{\tau}_{j-1}, \tilde{\tau}_j \rrbracket}) \cdot \big((X^{k,n})^{\tilde{\tau}_j} \mathbf{1}_{\rrbracket \tilde{\tau}_{j-1}, \infty \llbracket}\big) \\
&= \sum_{j,k,n=1}^{\infty} \big(H \hat{\mathbf{1}}_{\rrbracket \tau_{k-1} \vee \tilde{\tau}_{j-1}, \tau_k \wedge \tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(n)}\big) \\
&\quad \cdot \big((X - X^{+}_{\tau_{k-1}})^{\tau_k \wedge \tilde{\tau}_j} \hat{\mathbf{1}}_{\rrbracket \tau_{k-1} \vee \tilde{\tau}_{j-1}, \infty \llbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(n)}\big)
\end{aligned}
$$

$$= \sum_{j,k,n=1}^{\infty} \left(H\hat{\mathbf{1}}_{\rrbracket \tau_{k-1}\vee\tilde{\tau}_{j-1},\tau_k\wedge\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\Omega^{k,n})} + 0^{(n)}\right) \tag{A.1}$$

$$\cdot \left(X^{\tau_k\wedge\tilde{\tau}_j}\hat{\mathbf{1}}_{\llbracket \tau_{k-1}\vee\tilde{\tau}_{j-1},\infty \llbracket \cap (\mathbb{R}_+\times\Omega^{k,n})} + 0^{(n)}\right)$$

$$= \sum_{j,k,n=1}^{\infty} \left(H\hat{\mathbf{1}}_{\rrbracket \tau_{k-1}\vee\tilde{\tau}_{j-1},\tau_k\wedge\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{j,n})} + 0^{(n)}\right) \tag{A.2}$$

$$\cdot \left(X^{\tau_k\wedge\tilde{\tau}_j}\hat{\mathbf{1}}_{\llbracket \tau_{k-1}\vee\tilde{\tau}_{j-1},\infty \llbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{j,n})} + 0^{(n)}\right)$$

$$= \sum_{k,j,n=1}^{\infty} \left(H\hat{\mathbf{1}}_{\rrbracket \tau_{k-1}\vee\tilde{\tau}_{j-1},\tau_k\wedge\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{j,n})} + 0^{(n)}\right) \tag{A.3}$$

$$\cdot \left((X - X_{\tilde{\tau}_{j-1}})^{\tau_k\wedge\tilde{\tau}_j}\hat{\mathbf{1}}_{\rrbracket \tau_{k-1}\vee\tilde{\tau}_{j-1},\infty \llbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{j,n})} + 0^{(n)}\right),$$

$$= \sum_{k,j,n=1}^{\infty} (\tilde{H}^{j,n}\mathbf{1}_{\rrbracket \tau_{k-1},\tau_k \rrbracket}) \cdot \left((\tilde{X}^{j,n})^{\tau_k}\mathbf{1}_{\rrbracket \tau_{k-1},\infty \llbracket}\right)$$

$$= \sum_{k,j,n=1}^{\infty} \left((\tilde{H}^{j,n}\cdot\tilde{X}^{j,n})^{\tau_k} - (\tilde{H}^{j,n}\cdot\tilde{X}^{j,n})^{\tau_{k-1}}\right)$$

$$= \sum_{j,n=1}^{\infty} \left((\tilde{H}^{j,n}\cdot\tilde{X}^{j,n})\right). \tag{A.4}$$

The steps utilize only definitions and basic properties of stochastic analysis in $\mathbb{R}^n$. Equations (A.1) and (A.3) follow because the integrand is zero when the shift in the integrator takes effect. Equation (A.2) follows from

$$\rrbracket \tau_{k-1}\vee\tilde{\tau}_{j-1}, \tau_k\wedge\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\Omega^{k,n}) = \rrbracket \tau_{k-1}\vee\tilde{\tau}_{j-1}, \tau_k\wedge\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{j,n}).$$

To prove that $\tilde{X}^{j,n}$ is a semimartingale for all $j,n\in\mathbb{N}$, suppose the $\mathbb{R}^n$-valued simple predictable processes $(S^i)_{i\in\mathbb{N}}$ and S satisfy $\lim_{i\to\infty} S^i = S$, with the convergence being ucp (assumed throughout). For $j,n\in\mathbb{N}$, let

$$H^i := S^i\hat{\mathbf{1}}_{\rrbracket \tilde{\tau}_{j-1},\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{j,n})} + 0^{(N)},$$

so that $\lim_{i\to\infty} H^i = H := S\hat{\mathbf{1}}_{\rrbracket \tilde{\tau}_{j-1},\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{k,n})} + 0^{(N)}$. Since each S^i is $\mathbb{R}^n$-simple predictable, each $H^{k,m,i}$, formed by dissecting H^i as in (2.2), is $\mathbb{R}^m$-simple predictable, hence $H^i\in\mathcal{L}(X)$. By Proposition 2.4, $\lim_{i\to\infty} H^i\cdot X = H\cdot X$, and by (A.4), $H\cdot X = (S\hat{\mathbf{1}}_{\rrbracket \tilde{\tau}_{j-1},\tilde{\tau}_j \rrbracket \cap (\mathbb{R}_+\times\tilde{\Omega}^{j,n})} + 0^{(N)})\cdot X = S\cdot\tilde{X}^{j,n}$. Since $H^i\cdot X = S^i\cdot\tilde{X}^{j,n}$, this proves that $\lim_{i\to\infty} S^i\cdot\tilde{X}^{j,n} = S\cdot\tilde{X}^{j,n}$, and therefore $\tilde{X}^{j,n}$ is a semimartingale.

Equation (A.4) above shows that $H\in\mathcal{L}(X)$ implies that $H\in\tilde{\mathcal{L}}(X)$, and furthermore that $H\cdot X = \widetilde{H\cdot X}$. The reset sequences (τ_k) and $(\tilde{\tau}_k)$ are arbitrary, so $\mathcal{L}(X) = \tilde{\mathcal{L}}(X)$, and $H\cdot X$ is independent of the choice of reset sequence. $\square$

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